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Bounds for the functional spectrum of operators in a Hilbert space

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Abstract. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with the unit operators I_1 and I_2, respectively, and A_{jk} be bounded operators acting from $\mathcal{H}_j$ into $\mathcal{H}_k$ $(j,k=1,2)$. We consider the block operator matrices $A=(A_{jk})$ and $F(z)=\operatorname{diag}(\hat{K}_j(z)I_j)$, where $\hat{K}_1(z)$ and $\hat{K}_2(z)$ are scalar analytic functions. The set of all $z\in\mathbb{C}$, such that $F(z)-A$ is boundedly invertible is called the F-regular set of A. The complement of the F-regular set to the complex plane is called the F-spectrum (the functional spectrum) of A. It is shown that the notion of the functional spectrum enables us to investigate, from the unified point of view, various types of coupled systems, in particular, systems of integral, fractional differential, integro-differential and differential-difference equations. We derive a bound for the functional spectrum and discuss applications of the obtained bound to the stability of the considered systems.

1. Introduction and statement of the main result

In this paper we introduce and investigate the notion of the functional spectrum. To the best of our knowledge it was not considered in the available literature. Below we show that the notion of the functional spectrum enables us to investigate from the unified point of view various types of coupled systems of equations.

Let $\mathcal{H}_k$ $(k=1,2)$ be a separable Hilbert space with a scalar product $\langle .,.\rangle_{\mathcal{H}_k}$, the norm $\|.\|_{\mathcal{H}_k}=\sqrt{\langle .,.\rangle_{\mathcal{H}_k}}$ and unit operator $I_{\mathcal{H}_k}=I$ $(k=1,2)$. Let us introduce the Hilbert space $\mathcal{X}=\mathcal{H}_1\oplus\mathcal{H}_2$ endowed with the scalar product defined by $\langle g,h\rangle_{\mathcal{X}}=\langle g_1,h_1\rangle_{\mathcal{H}_1}+\langle g_2,h_2\rangle_{\mathcal{H}_2}$ and the norm $\|h\|_{\mathcal{X}}=(\|h_1\|^2_{\mathcal{H}_1}+\|h_2\|^2_{\mathcal{H}_2})^{1/2}$, $(g_k,h_k\in\mathcal{H}_k,\ k=1,2;\ g=(g_1,g_2),\ h=(h_1,h_2))$. For a linear operator B in $\mathcal{X}$, B^* is the adjoint one, B^{-1} is the inverse one, $\sigma(B)$ is the spectrum and $\|B\|=\|B\|_{\mathcal{X}}$ is the operator norm. In addition, $\mathcal{B}(\mathcal{H}_k,\mathcal{H}_j)$ $(j,k=1,2)$ denotes the set of linear bounded operators C acting from $\mathcal{H}_k$ into $\mathcal{H}_j$ with the norm $\|C\|=\|C\|_{jk}=\sup_{h\in\mathcal{H}_j}\|Ch\|_j/\|h\|_k$. To

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explain our motivation, consider the coupled system of integral equations

$$\begin{aligned} \int_0^t K_1(t-s)u_1(s)ds &= A_{11}u_1(t) + A_{12}u_2(t) + f_1(t), \\ \int_0^t K_2(t-s)u_2(s)ds &= A_{21}u_1(t) + A_{22}u_2(t) + f_2(t). \end{aligned} \tag{1}$$

Here and below $A_{jk} \in \mathcal{B}(H_k, H_j)$ $(j,k = 1,2)$, $K_j(t) : [0,\infty) \to \mathbb{R}$ are continuous real-valued functions admitting the Laplace transform, $t \geq 0$ and $f_j(t) : [0,\infty) \to \mathcal{H}_j$ $(j = 1,2)$ are continuous functions admitting the Laplace transform.

Denote the Laplace transform of a vector-valued function $v(t)$ $(t \geq 0)$ by $(\mathcal{L}v)(z)$, where $z \in \mathbb{C}$ is the dual variable, and put $\hat{K}_j(z) = (\mathcal{L}K_j)(z)$, $\hat{u}_j(z) = (\mathcal{L}u_j)(z)$, $\hat{f}_i(z) = (\mathcal{L}f_j)(z)$. Applying the Laplace transform to (1), we obtain

$$\begin{aligned} \hat{K}_1(z)\hat{u}_1(z) &= A_{11}\hat{u}_1(z) + A_{12}\hat{u}_2(z) + \hat{f}_1(z), \\ \hat{K}_2(z)\hat{u}_2(z) &= A_{21}\hat{u}_1(z) + A_{22}\hat{u}_2(z) + \hat{f}_2(z). \end{aligned} \tag{2}$$

Introduce the operator matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{3}$$

Then system (2) takes the form

$$(F(z) - A)\hat{v}(z) = \hat{h}(z), \tag{4}$$

where

$$F(z) = \begin{pmatrix} \hat{K}_1(z)I_1 & 0 \\ 0 & \hat{K}_2(z)I_2 \end{pmatrix},$$

$\hat{v}(z) = (\hat{u}_1(z), \hat{u}_2(z))$, $\hat{h}(z) = (\hat{f}_1(z), \hat{f}_2(z))$.

The set of all $z \in \mathbb{C}$, such that $F(z) - A$ is boundedly invertible in $\mathcal{X}$ will be called the F-regular set of A and will be denoted by $\rho_F(A)$. The complement of the F-regular set to the complex plane will be called the *F-spectrum (the functional spectrum)* of A and will be denoted by $\sigma_F(A)$.

According to (4), system (1) has a unique solution $v(t) = (u_1(t), u_1(t))$ provided $z \in \rho_F(A)$. Besides its Laplace transform is

$$\hat{v}(z) = (F(z) - A)^{-1}\hat{h}(z). \tag{5}$$

Now let us consider the coupled system of fractional order differential equations

$$D^{\gamma_1}u_1(t) = A_{11}u_1(t) + A_{12}u_2(t) + f_1(t),$$

$$D^{\gamma_2}u_2(t) = A_{21}u_1(t) + A_{22}u_2(t) + f_2(t), \quad 0 < \gamma_1,\ \gamma_2 < 1,\ t > 0, \tag{6}$$

where D^γ means the Riemann-Liouville fractional derivative of order γ ($0 < \gamma < 1$) [14]. Fractional-order systems arise in the control theory and many other applications, cf. [1, 4, 15]. Take the zero initial conditions

$$u_1(0) = u_2(0) = 0. \tag{7}$$

Since $u_j(0) = 0$, we have $(\mathcal{L}D^{\nu_j}u_j)(z) = z^{\nu_j}\hat{u}_j(z)$ [14, p. 122]. By the Laplace transform (6), (7) can be reduced to (2) with $\hat{K}_j(z) = z^{\gamma_j}$ $(j = 1, 2)$.

Similarly, one can consider the integro-differential system

$$\begin{aligned} \int_0^t K_1(t-s)u_1(s)ds &= A_{11}u_1(t) + A_{12}u_2(t) + f_1(t), \\ du_2(t)/dt &= A_{21}u_1(t) + A_{22}u_2(t) + f_2(t), \quad t \geq 0, \end{aligned} \tag{8}$$

with the initial condition

$$u_2(0) = 0, \tag{9}$$

where $K_1(t)$ is the same as above. In this case we have (2) with $\hat{K}_1(z) = (\mathcal{L}K_1)(z)$ and $\hat{K}_2(z) = z$.

An additional example of the matrix F gives us the coupled system of the difference and differential equations

$$u_1(t-\tau) = A_{11}u_1(t) + A_{12}u_2(t) + f_1(t) \;\; (0 \leq \tau = \text{const} < \infty), \tag{10a}$$

$$du_2(t)/dt = A_{21}u_1(t) + A_{22}u_2(t) + f_2(t), \quad t \geq 0, \tag{10b}$$

with the zero initial conditions

$$u_1(t) = 0, \quad -\tau \leq t \leq 0, \; u_2(0) = 0.$$

In this case we have (2) with $\hat{K}_1(z) = e^{-z\tau}$ and $\hat{K}_2(z) = z$. Our approach below complements the well-known results on the stability of differential-delay systems, cf. [6, 7, 12, 13].

As the above examples show, the functional spectrum arises in various problems.

The aim of this paper is to derive bounds for the F-spectrum, assuming that we have an information about $\sigma(A_{jj})$ and

$$\|(A_{jj} - \lambda I_j)^{-1}\|_{\mathcal{H}_j} \leq \phi_j(1/d(A_{jj}, \lambda)), \quad \lambda \notin \sigma(A_{jj}), \; j = 1, 2, \tag{11}$$

where

$$d(A_{jj}, \lambda) = \inf_{s \in \sigma(A_{jj})} |s - \lambda|,$$

i.e., the distance between a complex number λ and $\sigma(A_{jj})$, and $\phi_1(y), \phi_2(y)$ are continuous monotonically increasing non-negative functions of a non-negative variable y, such that $\phi_j(0) = 0$ and $\phi_j(\infty) = \infty$.

To formulate our main result, let denote by $r_j = r_j(q_j)$ the unique positive root of the equation

$$q_j\phi_j(1/x) = 1, \tag{12}$$

where q_j, $j = 1, 2$, are positive numbers, such that

$$q_1 q_2 = \|A_{12}\| \|A_{21}\|, \quad \|A_{jk}\| = \|A_{jk}\|_{jk}, \quad j, k = 1, 2.$$

One can take $q_1 = \|A_{12}\|$, $q_2 = \|A_{21}\|$, or

$$q_1 = (\|A_{12}\| \|A_{21}\|)^{\nu}, \quad q_2 = (\|A_{12}\| \|A_{21}\|)^{1-\nu}, \quad 0 \le \nu \le 1,$$

or one can choice q_1 from the condition

$$r_1(q_1) = r_2(q_2) = r_2(\|A_{12}\| \|A_{21}\|/q_1).$$

Below we suggest estimates for r_j in the concrete cases. Now we are in a position to formulate the main result of the paper.

Theorem 1. *Let A be defined by (3) with $A_{jk} \in \mathcal{B}(\mathcal{H}_k, \mathcal{H}_j)$ $(j, k = 1, 2)$ and conditions (11) hold. Then for any $\mu \in \sigma_F(A)$ we have either $d(A_{11}, \hat{K}_1(\mu)) \le r_1(q_1)$ or (and) $d(A_{22}, \hat{K}_2(\mu)) \le r_2(q_2)$.*

The proof of this theorem is presented in the next section.

For example, if A_{11} and A_{22} are normal: $A_{jj} A_{jj}^* = A_{jj}^* A_{jj}$, $j = 1, 2$, then, as is well-known,

$$\|(A_{jj} - \lambda I_j)^{-1}\| = \frac{1}{d(A_{jj}, \lambda)}, \quad \lambda \notin \sigma(A_{jj}),\ j = 1, 2.$$

So $\phi_j(y) = y$. Equation (12) takes the form $q_j/x = 1$ and consequently, $r_j(q_j) = q_j$.

Theorem 1 is sharp in the following sense. If A_{11} and A_{22} are arbitrary bounded operators and $\|A_{12}\| \|A_{21}\| = 0$, then, taking $q_1 = q_2 = \sqrt{\|A_{12}\| \|A_{21}\|} = 0$, we have $r_1 = r_2 = 0$ and due to Theorem 1, for any $\mu \in \sigma_F(A)$ there is an $s \in \sigma(A_{11}) \cup \sigma(A_{22})$, such that either $\hat{K}_1(\mu) = s$ or (and) $\hat{K}_2(\mu) = s$.

2. Proof of Theorem 1

We begin with the following result.

Lemma 1. *Let A_{11} and A_{22} be boundedly invertible in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let*

$$\|A_{22}^{-1} A_{21} A_{11}^{-1} A_{12}\|_{\mathcal{H}_2} < 1. \tag{13}$$

Then A is boundedly invertible in $\mathcal{X}$.

Proof. For an arbitrary $b = (b_1, b_2)$, $b_j \in \mathcal{H}_j$, consider the equation

$$Ax = b, \quad x = (x_1, x_2),\ x_j \in \mathcal{H}_j,\ j = 1, 2.$$

Then

$$A_{11} x_1 + A_{12} x_2 = b_1 \quad \text{and} \quad A_{21} x_1 + A_{22} x_2 = b_2$$

and therefore

$$x_1 = -A_{11}^{-1} A_{12} x_2 + A_{11}^{-1} b_1. \tag{14}$$

This gives

$$A_{21}(-A_{11}^{-1}A_{12}x_2 + A_{11}^{-1}b_1) + A_{22}x_2 = b_2$$

and consequently,

$$A_{22}^{-1}A_{21}(-A_{11}^{-1}A_{12}x_2 + A_{11}^{-1}b_1) + x_2 = A_{22}^{-1}b_2.$$

So

$$(-A_{22}^{-1}A_{21}A_{11}^{-1}A_{12} + I_2)x_2 = w,$$

where

$$w = A_{22}^{-1}b_2 + A_{22}^{-1}A_{21}A_{11}^{-1}b_1.$$

According to (13), $-A_{22}^{-1}A_{21}A_{11}^{-1}A_{12} + I_2$ is boundedly invertible and we get

$$x_2 = (-A_{22}^{-1}A_{21}A_{11}^{-1}A_{12} + I_2)^{-1}w.$$

This and (14) imply that A is boundedly invertible, as claimed. □

If

$$q_1\|(A_{11} - \hat{K}_1(z)I_1)^{-1}\| < 1$$

and

$$q_2\|(A_{22} - \hat{K}_2(z)I_2)^{-1}\| < 1,$$

then due to Lemma 1 $z \in \rho_F(A)$. Hence it follows.

Corollary 1. *Under the hypothesis of Theorem 1 any $\mu \in \sigma_F(A)$ satisfies at least to one of the following inequalities:*

$$q_1\|(A_{11} - \hat{K}_(\mu)I_1)^{-1}\| \geq 1$$

and

$$q_2\|(A_{22} - \hat{K}_2(\mu)I_2)^{-1}\| \geq 1.$$

Proof of Theorem 1. From (11) and Corollary 1 it follows that for any $\mu \in \sigma_F(A)$, either

$$q_1\phi_1(1/d(A_{11}, \hat{K}_1(\mu)) \geq 1, \tag{15}$$

or (and)

$$q_2\phi_2(1/d(A_{22}, \hat{K}_2(\mu)) \geq 1. \tag{16}$$

Hence, either $d(A_{11}, \hat{K}_1(\mu)) \leq r_1$, or (and) $d(A_{22}, \hat{K}_2(\mu)) \leq r_2$. This proves the theorem. □

3. Functional spectrum of finite matrices

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space with a scalar product $\langle .,. \rangle_n$, the norm $\|.\|_n = \sqrt{\langle .,. \rangle_n}$ and unit matrix I_n. For integers n_1 and n_2, $\mathbb{C}^{n_1 \times n_2}$ is the set of complex $n_1 \times n_2$-matrices. In this section we take $\mathcal{H}_j = \mathbb{C}^{n_j}$ and $A_{jk} \in \mathbb{C}^{n_j \times n_k}$ $(j,k = 1,2)$ with $n_1 + n_2 = n$.

The following quantity (the departure from normality of $A \in \mathbb{C}^{n \times n}$) plays an essential role in this section:

$$g(A) = \left[N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2 \right]^{1/2},$$

where $N_2(A) = (\text{tr } AA^*)^{1/2}$ is the Frobenius (Hilbert-Schmidt) norm of A, $\lambda_k(A)$ $(k = 1, \ldots, n)$ are the eigenvalues of A taken with their multiplicities. The following relations are checked in [8, Section 2.1] (see also [9, Section 3.1]).

$$g^2(A) \le N_2^2(A) - \text{tr } (A^2)| \quad \text{and} \quad g^2(A) \le N_2^2(A - A^*)/2. \tag{17}$$

If A is a normal matrix: $AA^* = A^*A$, then $g(A) = 0$. If A_1 and A_2 are commuting matrices, then $g(A_1 + A_2) \le g(A_1) + g(A_2)$.

Thanks to Corollary 2.1.2 [8] (see also [9, Theorem 3.2]), the resolvent $R_\lambda(A) = (A - \lambda I)^{-1}$ of a linear operator A in $\mathbb{C}^n$ satisfies the inequality

$$\|(A - I\lambda)^{-1}\|_{\mathbb{C}^n} \le \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}(d(A,\lambda))^{k+1}} \quad (\lambda \notin \sigma(A)).$$

Recall that $d(A, \lambda) = \min_{k=1,\ldots,n} |\lambda - \lambda_k(A)|$.

We thus can write

$$\|(A_{jj} - I_j \hat{K}_j(\lambda))^{-1}\|_{\mathbb{C}^{n_j}} \le \sum_{k=0}^{n_j - 1} \frac{g^k(A_{jj})}{\sqrt{k!} d^{k+1}(A_{jj}, \hat{K}_j(\lambda))}, \quad \hat{K}_j(\lambda) \notin \sigma(A_{jj}),$$

with

$$d(A_{jj}, \hat{K}_j(\lambda)) = \min_{k=1,\ldots,n_j} |\hat{K}_j(\lambda) - \lambda_k(A_{jj})|.$$

So inequality (11) holds with

$$\phi_j(y) = \sum_{k=0}^{n_j - 1} \frac{g^k(A_{jj}) y^{k+1}}{\sqrt{k!}}.$$

Equation (12) takes the form

$$q_j \sum_{k=0}^{n_j - 1} \frac{g^k(A_{jj})}{x^{k+1}\sqrt{k!}} = 1.$$

This equation is equivalent to the following algebraic one:

$$x^{n_j} = q_j \sum_{k=0}^{n_j-1} \frac{x^{n_j-k-1} g^k(A_{jj})}{\sqrt{k!}}. \tag{18}$$

So in this case $r_j = r_{n_j}$, where r_{n_j} is the unique positive root of (18). Put

$$\eta_j = q_j \phi_j(1) = q_j \sum_{k=0}^{n_j-1} \frac{g^k(A_{jj})}{\sqrt{k!}}.$$

Due to the simple [8, Lemma 1.6.1] $r_{n_j} \le \delta_{n_j}$, where

$$\delta_{n_j} = \begin{cases} (\eta_j)^{1/n_j}, & \text{if } \eta_j \le 1; \\ \eta_j & \text{if } \eta_j \ge 1. \end{cases}$$

Now Theorem 1 yields.

Corollary 2. *Let A be defined by* (3) *with $A_{jk} \in \mathbb{C}^{n_j \times n_k}$, $j, k = 1, 2$. Then for any $\mu \in \sigma_F(A)$ we have either $d(A_{11}, \hat{K}_j(\mu)) \le r_{n_1}(q_1) \le \delta_{n_1}$ or (and) $d(A_{22}, \hat{K}_j(\mu)) \le r_{n_2}(q_2) \le \delta_{n_2}$.*

4. Functional spectrum of operators with Hilbert-Schmidt components

Let $\mathcal{S}_2$ be the ideal of Hilber-Schmidt operators K with the finite norm $N_2(K) := (\text{trace} KK^*)^{1/2}$. In this section it is supposed that

$$A_{jk} \in \mathcal{B}(\mathcal{H}_k, \mathcal{H}_j) \ \text{ and } \ \text{Im } A_{jj} := (A_{jj} - A_{jj}^*)/(2i) \in \mathcal{S}_2, \ \ j, k = 1, 2. \tag{19}$$

Numerous integral operators satisfy these conditions. Introduce the quantity

$$g_I(A_{jj}) := \sqrt{2}[N_2^2(\text{Im } A_{jj}) - \sum_{k=1}^{\infty} (\text{Im } \lambda_k(A_{jj}))^2]^{1/2}.$$

Obviously,

$$g_I(A_{jj}) \le \sqrt{2} N_2(\text{Im } A_{jj}). \tag{20}$$

Due to [9, Theorem 9.1], under condition (19) we have

$$\|(A_{jj} - \lambda I_j)^{-1}\| \le \frac{\sqrt{e}}{d(A_{jj}, \lambda)} \exp\left[\frac{g_I^2(A_{jj})}{2d^2(A_{jj}, \lambda)} \right] \ \ (\lambda \notin \sigma(A_{jj})). \tag{21}$$

Thus, (11) holds with

$$\phi_j(y) = y \exp\left[\tfrac{1}{2}(1 + y^2 g_I^2(A_{jj})) \right], \quad y > 0.$$

Note that some other norm estimates for resolvents can be found in [8, 9, 11].

Equation (12) takes the form

$$q_j \frac{\sqrt{e}}{x} \exp\left[\frac{v_j^2}{2x^2} \right] = 1, \ \text{where } v_j = \frac{g_I(A_{jj})}{\sqrt{2}}, \ x > 0. \tag{22}$$

Now, Theorem 1 yields.

Corollary 3. *Let A be defined by* (3) *and conditions* (19) *hold. Then for any $\mu \in \sigma_F(A)$ we have either $d(A_{11}, \hat{K}_1(\mu)) \le \hat{r}_1(q_1)$ or (and) $d(A_{22}, \hat{K}_2(\mu)) \le \hat{r}_2(q_2)$, where $\hat{r}_j(q_j)$ $(j = 1, 2)$ is the unique positive root of equation* (22).

From (22) with $y = \frac{v_j}{x}$ we get

$$\frac{q_j\sqrt{e}}{v_j} y \exp\,[y^2] = 1. \tag{23}$$

To estimate the unique positive root of this equation we can apply the following result.

Lemma 2 ([9, Lemma 7.2])**.** *For any integer $p \ge 1$, the unique positive root z_a of the equation*

$$y \exp\,[(y+1)^p] = a, \quad a = const > 0$$

satisfies the inequality $z_a \ge \delta_p(a)$, where

$$\delta_p(a) := \begin{cases} 2^{-1/p}(\ln\, a + 2^p)^{1/p} - 1, & \textit{if } a \ge \exp\,[2^p]; \\ \exp\,[-2^p]a, & \textit{if } a < \exp\,[2^p]. \end{cases}$$

This lemma with $a = \hat{a}_j$, where

$$\hat{a}_j = \frac{v_j}{q_j\sqrt{e}}$$

implies that the unique positive root $\hat{y}_j$ of the equation (23) satisfies the inequality $\hat{y}_j \ge \psi_j$, where

$$\psi_j := \begin{cases} 2^{-1/2}(\ln\, \hat{a}_j + 4)^{1/2} - 1, & \text{if } \hat{a}_j \ge e^4; \\ \exp\,[-4]\hat{a}_j, & \text{if } \hat{a}_j < e^4. \end{cases}$$

Consequently

$$\hat{r}_j(q_j) \le \frac{v_j}{\hat{y}_j} \le \frac{v_j}{\psi_j}.$$

5. L^2-INPUT-OUTPUT STABILITY

Let $L^2(\mathbb{R}, \mathcal{X})$ be the space of functions $w(.)$ defined on $(-\infty, \infty)$ with values in $\mathcal{X}$ and the norm

$$\|w\|_{L^2(\mathbb{R})} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \|w(y)\|^2_{\mathcal{X}} dy\right)^{1/2},$$

and let $L^2(R_+, \mathcal{X})$ be the space of functions $f(.)$ defined on $R_+ = [0, \infty)$ with values in $\mathcal{X}$ and the norm

$$\|f\|_{L^2(R_+)} = \left(\int_0^{\infty} \|f(t)\|^2_{\mathcal{X}} dt\right)^{1/2}.$$

A solution of (1) is a function defined on $[0,\infty)$ with values in $\mathcal{X}$, and satisfying (1).

System (1) *is said to be L^2-input-output stable (L^2-IO-stable)*, if for any $f \in L^2(R_+, \mathcal{X})$, it has a unique solution v admitting the Laplace transform and

$$\|v\|_{L^2(R_+)} \le c_0 \|f\|_{L^2(R_+)},$$

where the constant c_0 is independent of f.

Assume the existence and uniqueness of solutions. Let $\hat{f} \in L^2(\mathbb{R}, \mathcal{X})$ and

$$\sup_{y\in\mathbb{R}} \|(A - F(iy))^{-1}\| < \infty. \tag{24}$$

Then, from (5) we have

$$\|\hat{v}\|_{L^2(\mathbb{R})} \le \sup_{y\in\mathbb{R}} \|(A - F(iy))^{-1}\| \|\hat{f}\|_{L^2(\mathbb{R})}.$$

Making use of the Parseval equality, we can assert that (24) provides the L^2-IO-stability of (1).

If for any $\mu \in \sigma_F(A)$, one has $\operatorname{Re} \hat{K}_j(\mu) < 0$, $j = 1, 2$, then by the Banach-Steinhaus theorem inequality (24) holds. But due to Theorem 1,

$$\operatorname{Re} \hat{K}_j(\mu) \le \alpha_j + r_j, \quad j = 1, 2, \text{ where } \alpha_j = \sup_{s\in\sigma(A_{jj}} \operatorname{Re} s. \tag{25}$$

Hence, we arrive at.

Corollary 4. *Let* $\max\{\alpha_1 + r_1, \alpha_1 + r_2\} < 0$. *Then,* (1) *is L^2-IO-stable.*

Similarly, we can investigate various systems, which can be reduced to (2). About the recent results on the input-stability see for instance [13] and the references given therein.

6. EXAMPLE

Let $\mathcal{H}_j = L^2(0, 1)$ be the Hilbert space of complex valued functions defined on $(0, 1)$ with the scalar product

$$\int_0^1 f(x)\overline{f}_1(x)dx, \quad f, f_1 \in L^2(0, 1),$$

and A be defined by (3) with arbitrary bounded operators A_{12} and A_{21}, and operators A_{jj} defined by

$$(A_{jj}f)(x) = c_j(x)f(x) + (V_j f)(x), \quad j = 1, 2;\ 0 \le x \le 1,$$

where $c_j(x)$ $(0 \le x \le 1)$ are real bounded measurable functions,

$$(V_j f)(x) = \int_0^x M_j(x, s)f(s)ds, \quad f \in L^2(0, 1).$$

Here $M_j(x,s)$ are complex functions defined on the set $\{0 \le s \le x \le 1\}$ and satisfying the condition

$$N_2^2(V_j) = \int_0^1 \int_0^x |M_j(x,s)|^2 ds, \quad dx < \infty.$$

Due to [9, Lemma 9.2],

$$N_2^2(V_j) = 2N_2^2(\mathrm{ImV_j}).$$

Let define S_j by $(S_j f)(x) = c_j(x) f(x)$. Thanks to [10, Lemma 7.1] A_{jj} is a $\hat{P}_s$-triangular operator (see [10, Definition 8.1]). Hence, according to [10, Corollary 3.4] we have

$$\sigma(A_{jj}) = \sigma(C_j) = \{s \in \mathbb{R} : s = c(x),\ 0 \le x \le 1\}.$$

Since Im $A_{jj} =$ Im V_j, we can write

$$N_2(\mathrm{Im} A_{jj}) = N_2(\mathrm{Im}\ V_j) = N_2(V_j)/\sqrt{2}.$$

Now, we can apply Corollary 4. About the recent results on the stability of integro-differential equations see, for instance [2, 3, 5] and the references given therein.

7. Concluding remarks

In this paper we introduced a new notion, the so-called functional spectrum. It generalizes the notion of the traditional spectrum. As the examples show, it allows rather easily to establish stability conditions for various types of coupled systems of equations in the finite-dimensional and infinite-dimensional Hilbert spaces. We have established conditions for the input-output stability. It will be interesting to apply the approach suggested in this paper to derive conditions for the exponential and asymptotic stabilities. Moreover, we have restricted ourselves by equations with bounded operators. It will be very useful to extend our result to equations with unbounded operators in Hilbert and Banach spaces.

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